Strength analysis of random short-fibre-reinforced metal matrix composite materials

Y. T. ZHU, G. ZONG, A. MANTHIRAM, Z. ELIEZER
Center for Materials Science and Engineering, The University of Texas at Austin, Austin, TX 78712, USA

A theory to analyse the strength of composite materials with randomly oriented short fibres has been developed. The short fibres are assumed to be uniformly distributed and randomly oriented in three dimensions. The non-homogeneous deformation within the composite has been taken into account in the strength calculation. The influences of thermal stress in the short fibres, the short-fibre dispersion hardening and the dislocation density in the matrix on the composite strength have all been estimated, and the strengthening mechanisms involved are discussed. A comparison with previous strength theory suggests that the present theory gives a better agreement with experimental data, and can be used to explain some experimental phenomena that remain unsolved.

1. Introduction

Composites reinforced with randomly oriented short fibres have become increasingly popular in recent years [1, 2]. The strength of such composites is one of the most important properties and it has attracted the attention of researchers in the composite area. In the design of a randomly oriented short-fibre-reinforced composite, it is essential to understand the strengthening mechanisms and the relationship between the strength of composites and the properties of its components. Although successful theories have been developed to predict the strength of composites having continuous or discontinuous fibres with unidirectional orientation [3–6], surprisingly only a limited amount of theoretical work has been done in understanding the strength of the composite with randomly oriented short fibres. Among the few models available in the literature, the models of both Chen [7] as well as Halpin and Kardos [8] treated the composite as a stack of unidirectional short-fibre-reinforced laminae bonded together at different angles, which is hardly true in reality. Also, these two theories do not provide any clear relationship between the composite strength and the component properties because they rely on the experimental failure strength and strain data of the unidirectional laminae.

Fukuda *et al.* [9] developed a theory to predict the strength of a composite reinforced with randomly oriented short fibres. Their theory considers neither the deformation inhomogeneity between the fibres and matrix of short fibre composites, nor the effect of matrix properties on the composite strength. As a result, the composite strength predicted by their theory is far below the experimental strength.

Friend [1, 10] proposed an empirical strength equation for randomly oriented short-fibre-reinforced metal matrix composites. Although his equation seems to agree with experimental data of some aluminium alloy matrix composites, it cannot explain the high strength of the composite with a pure aluminium matrix. This is because Friend's theory lacks identification of the dominant matrix strengthening mechanism.

The objective of the present work was (1) to develop a new strength theory for three-dimensional randomly oriented short-fibre-reinforced composite materials, which will avoid the shortcomings of previous theories; and (2) to compare and evaluate all the existing strengthening mechanisms of the metal matrix composites reinforced with short fibres.

2. Strength analysis

The random short-fibre-reinforced composites are different from the continuous and particulate composites in that both the matrix and fibres of short-fibre composites carry load. Compared to that of continuous fibre composites, the matrix of random short-fibre-reinforced metal matrix composite is subjected to work hardening and larger strain than the fibre. In addition to directly bearing the load, the short fibres will also have dispersion strengthening effect on the matrix in a similar way as does the particulate.

Owing to thermal expansion mismatch between the fibre and the matrix, compressive thermal stress will develop in the fibre and a high dislocation density will result in the metal matrix as the composite is cooled from the high synthesis temperature [11, 12]. The compressive thermal stress in the fibres and high dislocation density in the matrix will contribute towards the composite strength.

During the tensile loading, off-axis fibres will tend to align with the loading direction due to the sample

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elongation along the loading direction and the area reduction perpendicular to the loading direction. This alignment process will have a negligible effect on the composite strength [13] and therefore will not be discussed here. All other strengthening mechanisms mentioned above will be analysed.

2.1. Direct short-fibre strengthening

The basic steps for deriving the direct strengthening by the randomly oriented short fibres are as follows. First, the load carried by an off-axis fibre at composite failure will be derived as a function of the off-axis angle, θ, which is the angle between the tensile loading direction and the longitudinal direction of the fibre. The second step is to obtain the average load contribution per fibre towards the loading direction. The third step is to calculate the effective number of fibres that is intercepted by a plane perpendicular to the loading direction. The total load carried by fibres at composite failure can then be calculated.

For simplicity, the following assumptions are made: (1) all fibres in the composite have the same tensile strength. This may not be true in reality, but it is a reasonable assumption and has been widely accepted in many other theories; (2) all the fibres have the same length and are randomly oriented, and (3) strong bonding between the fibres and the matrix.

To obtain the load carried by an off-axis fibre as a function of the off-axis angle, θ, the strain in the fibre has to be calculated first. Fig. 1 shows a fibre with a off-axis angle $0 \le \theta \le \pi/2$ in a composite sample. With x_3 as the loading direction, the composite failure strain in the x_3 direction is

$$\varepsilon_{33} = \varepsilon_c \tag{1}$$

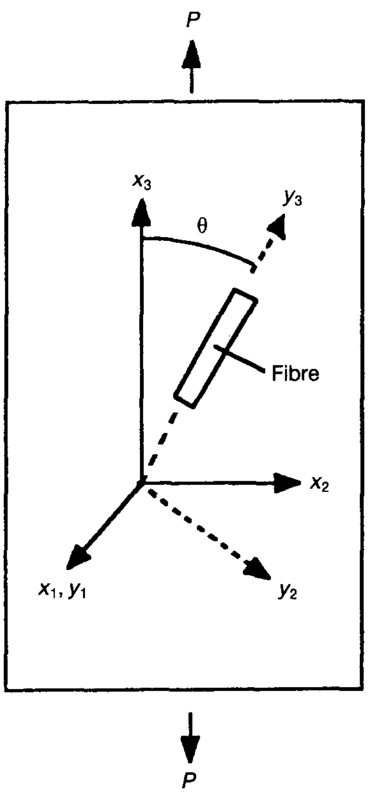

Figure 1 Definition of the off-axis angle, θ.

where ε_c is the composite failure strain under tensile load. The strain in the x_1 and x_2 directions are

$$\varepsilon_{11} = \varepsilon_{22} = -\nu\varepsilon_{33} \tag{2}$$

where ν is the Poisson's ratio of the composite.

To calculate the strain in the off-axis fibre, let us rotate the coordinate system around the x_1 axis clockwise for an angle of θ (see Fig. 1). The transformation matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{3}$$

where $a_{ij} = \cos\alpha_{ij}$, and α_{ij} is the angle between y_i and x_j. The strain in y_3 direction (along the off-axis fibre) can be calculated as

$$\begin{aligned} \varepsilon^{y}_{33}(\theta) &= \sum_{i=1}^{3} a_{3i} \sum_{j=1}^{3} a_{3j}\varepsilon_{ij} \\ &= \varepsilon_{33}(\cos^2\theta - \nu\sin^2\theta) \end{aligned} \tag{4}$$

Substituting Equation 1 into Equation 4 yields

$$\varepsilon^{y}_{33}(\theta) = \varepsilon_c(\cos^2\theta - \nu\sin^2\theta) \tag{5}$$

In the short-fibre composite, the matrix is also a load-bearing component. It will undergo higher deformation because it has a lower Young's modulus than the fibre. As a result, the actual composite failure strain, ε_c, is usually larger than the fibre failure strain, ε_f, due to the inhomogeneous deformation. Setting $\varepsilon^{y}_{33}(\theta) = \varepsilon_f$ in Equation 5 and solving the equation yields

$$\theta_0 = \cos^{-1}\left(\frac{k + \nu}{1 + \nu}\right)^{1/2} \tag{6}$$

where ε_f is the fibre failure strain and

$$k = \varepsilon_f/\varepsilon_c \tag{7}$$

At $\theta < \theta_0$, $\varepsilon^{y}_{33}(\theta)$ calculated with Equation 5 will be larger than ε_f. Let us assume that all the fibres with off-axis angle less than θ_0 fail simultaneously and their failure causes immediate composite failure. Then, at composite failure, the load carried by those fibres with $\theta < \theta_0$ will be

$$f(\theta) = a_f\sigma_f = f_0 \tag{8}$$

where a_f is the fibre cross-sectional area, σ_f is the fibre strength and f_0 is the maximum load a fibre can carry.

The strain in the fibres with $\theta > \theta_0$ can be considered approximately equal to $\varepsilon^{y}_{33}(\theta)$. The load carried by those fibres, therefore, can be calculated as

$$f(\theta) = E_f a_f \varepsilon^{y}_{33}(\theta) \tag{9}$$

where E_f is the Young's modulus of the fibre. Considering $\sigma_f = E_f\varepsilon_f$ and substituting Equations 5, 7 and 8 into Equation 9, we get

$$f(\theta) = \frac{f_0}{k}(\cos^2\theta - \nu\sin^2\theta) \tag{10}$$

Setting $f(\theta) = 0$ in Equation 10 yields

$$\sin^2\theta_f = 1/(1 + \nu) \tag{11}$$

where θ_f can be calculated from Equation 11. From Equation 10, it can be seen that $f(\theta)$ is positive if $\theta < \theta_f$, which means tensile load in the fibre. But, if θ is larger than θ_f, $f(\theta)$ will be negative, which means compressive stress in the fibre. Because the fibre has higher Young's modulus than the matrix, the fibre should always have a higher resistance to deformation than the matrix does, regardless of its stress condition. Therefore, those fibres under compressive stress will also make a positive contribution toward the composite strength. Based on the above argument, the absolute value of $f(\theta)$ should be used for the calculation of average load contribution per fibre towards the loading direction.

With the above discussion, the load carried by fibres with different off-axis angles can be summarized as

$$f(\theta) = \begin{cases} f_0 & \theta \leqslant \theta_0 \\ \dfrac{f_0}{k}(\cos^2\theta - \nu\sin^2\theta) & \theta_0 < \theta \leqslant \theta_f \\ -\dfrac{f_0}{k}(\cos^2\theta - \nu\sin^2\theta) & \theta_f < \theta \leqslant \dfrac{\pi}{2} \end{cases} \quad (12)$$

To derive the average load contribution per fibre towards the loading direction, the fibre orientation distribution at composite failure as a function of off-axis angle, θ, is also needed. It is assumed that the fibres are completely randomly oriented before loading. Then, the fibre distribution can be expressed as [9]

$$N(\theta') = N\sin\theta' \quad (13)$$

where N is the total number of fibres in the specimen. The average load contribution per fibre towards the loading direction, $\bar{f}_{\parallel}$, can be calculated as

$$\bar{f}_{\parallel} = \frac{1}{N}\int_0^{\pi/2} N(\theta)|f(\theta)|\cos(\theta)\,d\theta \quad (14)$$

Substituting Equations 12 and 13 into Equation 14 and integrating Equation 14, we get

$$\bar{f}_{\parallel} = \frac{a_f\sigma_f}{4}\frac{1+\nu^2}{1+\nu} \quad (15)$$

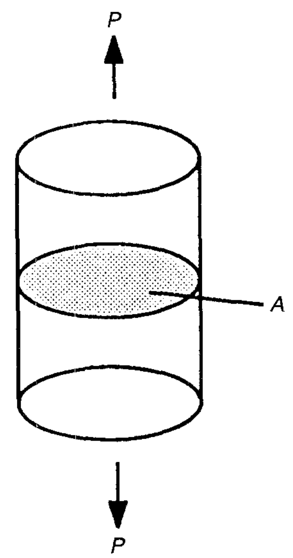

Figure 2 A composite sample and its cross-section.

Assume that A is the sample cross-sectional area perpendicular to the loading direction (see Fig. 2) and N_e is the effective number of fibres that is intercepted by the cross-section. Owing to the critical load transfer length of short fibres, not every fibre cut by the cross-section is load bearing at the cutting point. N_e represents the number of fibres that is load bearing at the cutting point. N_e can be expressed as (see Appendix)

$$N_e = \frac{AV_f}{2a_f}\left(1 - \frac{l_c}{2l}\right) \quad (16)$$

where V_f is volume fraction of the short fibres, l_c is the critical load transfer length, and l is the fibre length. Therefore, at the cross-section, the total load carried by the short fibres at composite failure is

$$P_f = N_e\bar{f}_{\parallel} \quad (17)$$

and the contribution of short fibres towards the composite strength is

$$\sigma_c^f = P_f/A \quad (18)$$

Substituting Equations 15–17 into Equation 18 yields

$$\sigma_c^f = \frac{1}{8}V_f\sigma_f\frac{1+\nu^2}{1+\nu}\left(1 - \frac{l_c}{2l}\right) \quad (19)$$

2.2. Dispersion hardening by short fibres

It can be understood that the dispersion strengthening by the random short fibres should be different from that by particulate. Therefore, the analysis used to calculate the dispersion strengthening for particulate composites is not appropriate for random short-fibre composites. Because no theory is available for short-fibre dispersion strengthening, it is necessary to develop a model for the calculation of random short-fibre dispersion strengthening.

Assume that a cube with edge length b is cut out of a random short-fibre-reinforced composite. The total number of short fibres in the cube can be calculated as

$$N_t = \frac{b^3V_f}{l\pi d^2/4} \quad (20)$$

where V_f is the fibre volume fraction, l is the fibre length and d is the fibre diameter.

The large cube can be considered to consist of N_c small cubes with edge length c, where N_c is determined in such a way that, on average, only one fibre passes through or is present in each small cube. This is equivalent to each small cube containing effectively only one fibre on average, and there are in total N_c effective fibres. Each effective fibre in a small cube can be treated approximately as a particle located at the centre of a small cube. Then, the distance between two adjacent particles should be equal to the small cube edge length, c. The value of c can be calculated as

$$c = \left(\frac{b^3}{N_c}\right)^{1/3} = \frac{b}{N_c^{1/3}} \quad (21)$$

In a short-fibre composite, the length of a short fibre may be several times that of the small cube edge. In other words, a short fibre may go through several, say n, small cubes. Therefore, a short fibre may be considered equal to n effective fibres, where n can be calculated as

$$n = l/c \tag{22}$$

The total number of effective short fibres is

$$N_c = N_t n \tag{23}$$

Substituting Equations 20–22 into Equation 23 and rearranging, we get

$$N_c = \left(\frac{4V_f b^2}{\pi d^2}\right)^{3/2} \tag{24}$$

Substituting Equation 24 into Equation 21 yields

$$c = \frac{d}{2}\left(\frac{\pi}{V_f}\right)^{1/2} \tag{25}$$

The segment of fibre in a small cube can be approximated as a spherical particle with the same volume. The volume of a fibre can be calculated as

$$V = \frac{1}{4}\pi d^2 c \tag{26}$$

and the volume of the particle is

$$V = \frac{4}{3}\pi r^3 \tag{27}$$

where r is the radius of the particle. Substituting Equations 25 and 26 into Equation 27 and solving the equation for r yields

$$r = \frac{d}{2}\left(\frac{9\pi}{16V_f}\right)^{1/6} \tag{28}$$

The mean free dislocation length in the matrix can be calculated as

$$\bar{l} = c - 2r \tag{29}$$

Substituting Equations 25 and 28 into Equation 29 yields

$$\bar{l} = \frac{d}{2}\left(\frac{\pi}{V_f}\right)\left[\left(\frac{\pi}{V_f}\right)^{1/3} - 6^{1/3}\right] \tag{30}$$

The additional shear stress required to move a dislocation due to dispersion strengthening can be calculated as [11]

$$\Delta\tau_{ml} = \frac{Gb}{\bar{l}} \tag{31}$$

where G is the shear modulus and $\boldsymbol{b}$ is the Burgers vector. Under unidirectional tension load, the additional normal stress needed to provide $\Delta\tau$ is [11]

$$\Delta\sigma_{ml} = 2\Delta\tau_{ml} \tag{32}$$

The above $\Delta\sigma$ is actually the dispersion strengthening. Substituting Equations 30 and 31 into Equation 32 yields

$$\Delta\sigma_{ml} = 4G\boldsymbol{b}\left\{d\left(\frac{\pi}{V_f}\right)^{1/6}\left[\left(\frac{\pi}{V_f}\right)^{1/3} - 6^{1/3}\right]\right\}^{-1} \tag{33}$$

Equation 33 is the final equation for calculating the dispersion strengthening in random short-fibre composites. It can be seen from Equation 33 that the dispersion strengthening is inversely proportional to the fibre diameter, d.

2.3. Thermal stress and dislocation strengthening

Metal matrix composites are generally synthesized at high temperatures. The difference in the thermal expansion coefficients between the fibre and the matrix can lead to the development of thermal stress and strain during the subsequent cooling of the composite from the high synthesis temperature [11]. High dislocation density may be introduced at the matrix–fibre interface due to the thermal strain, which will strengthen the matrix [12]. In addition, residual thermal compressive stress may also exist in the fibre, which will increase the apparent strength of the fibre. All of these must be taken into account in calculating the composite strength.

The dislocation strengthening in term of shear stress can be calculated as [11]

$$\Delta\tau_{m2} = \alpha G\boldsymbol{b}\rho^{1/2} \tag{34}$$

where $\alpha = 0.3$–0.5 is a constant, G is the shear modulus, $\boldsymbol{b}$ is Burgers vector and ρ is the dislocation density. The corresponding normal stress needed to overcome $\Delta\tau_{m2}$ during tensile test is [11]

$$\Delta\sigma_{m2} = 2\Delta\tau_{m2} = 2\alpha G\boldsymbol{b}\rho^{1/2} \tag{35}$$

It is very difficult to calculate quantitatively the dislocation density in the matrix and the compressive stress in the fibre because of their dependence on the thermal processing history of the composite. Nevertheless, qualitative estimation can be made based on experimental data. Arsenault and Fisher [12] observed that the dislocation density in the matrix of an SiC fibre-reinforced aluminium alloy composite is about 1×10^{10} to 4×10^{10} cm^{-2} and the residual compressive stress can be up to 35 MPa. Pure aluminium matrix may have a dislocation density much larger than the above observed value because of its lower yield strength.

2.4. Composite strength calculation

With the analysis of above strengthening mechanisms, the composite strength can be calculated as

$$\sigma_c = (1 - V_f)\sigma'_m + \frac{V_f\sigma'_f}{8}\frac{1 + \nu^2}{1 + \nu}\left(1 - \frac{l_c}{2l}\right) \tag{36}$$

where V_f is the fibre volume fraction, σ'_m is the matrix strength at composite failure, σ'_f is the apparent fibre strength, ν is the Poisson's ratio, l_c is the critical load transfer length and l is the fibre length. σ'_m can be calculated as

$$\sigma'_m = \sigma_m + \Delta\sigma_{ml} + \Delta\sigma_{m2} \tag{37}$$

where σ_m is the calculated matrix stress at composite failure without the consideration of dispersion hardening and dislocation strengthening. Substitution of Equations 33 and 35 into Equation 37 yields

$$\sigma'_m = \sigma_m + 4Gb\left\{d\left(\frac{\pi}{V_f}\right)^{1/6}\left[\left(\frac{\pi}{V_f}\right)^{1/3} - 6^{1/3}\right]\right\}^{-1} + 2\alpha Gb\rho^{1/2} \quad (38)$$

σ'_f can be calculated as

$$\sigma'_f = \sigma_f + \sigma_r \quad (39)$$

where σ_f is the fibre strength and σ_r is the residual compressive stress in the fibre.

3. Discussion

The strength of randomly oriented short-fibre-reinforced metal matrix composite can be estimated using Equations 36, 38 and 39. Although quantitative calculation may be difficult because the dislocation density in the matrix and the residual compressive stress in the fibres is usually unknown, this analysis does give us some information about the significance of each strengthening mechanism.

Friend [1, 10] developed an empirical strength equation for randomly oriented short-fibre-reinforced metal matrix composites. The equation takes the form

$$\sigma_c = \sigma_m(1 - V_f) + \frac{V_f\sigma_f}{5}\left(1 - \frac{l_c}{2l}\right) \quad (40)$$

Although this equation can predict the strength of the δ-alumina fibre-reinforced aluminium alloy composites very well, it cannot explain the high strength of the pure aluminium matrix composite. For example, Friend's equation predicts a strength of approximately 120 MPa for a commercially pure aluminium matrix composite with 25% δ-alumina fibre. The predicted strength is 55 MPa less than the experimentally measured strength, 175 MPa. Referring to Arsenault and Fisher's work [12], we may reasonably assume that the matrix dislocation density, ρ, is $3 \times 10\ \mathrm{cm}^{-2}$ and the residual compressive stress in the fibre is 35 MPa. Our analysis using Equation 36 predicts a strength value of 175 MPa, which agrees with the experimental data. Another example is a pure aluminium matrix composite with 20% δ-alumina fibre. Friend's equation predicts its strength as approximately 75 MPa, which is 65 MPa less than the experimentally measured strength, 140 MPa. Assuming that the matrix dislocation density is $2 \times 10\ \mathrm{cm}^{-2}$ because of the lower fibre volume fraction in this case, our analysis predicts its strength value as 140 MPa, which again agrees with the experimental data.

The present analysis can also explain the strength of the aluminium alloy matrix composite very well. For example, an Al–7Si matrix composite with 20% δ-alumina fibre has an experimental strength of 237 MPa [1]. Assuming that the residual compressive stress is 35 MPa and the matrix dislocation density is $10\ \mathrm{cm}^{-2}$ (aluminium alloy has a higher yielding strength and thus has a lower dislocation density than pure aluminium after higher temperature synthesis), the present analysis predicts its strength to be 244 MPa. Of course, all the predictions by the present analysis are only qualitative because the dislocation density and residual fibre stress used in the calculation are not experimentally determined values. Nevertheless, the present analysis give us some idea about the strengthening mechanisms in randomly oriented short-fibre-reinforced metal matrix composites. It can also provide composite engineers with some approaches to improve the composite strength.

4. Conclusions

The strength analysis for randomly oriented short-fibre-reinforced metal matrix composites developed in this paper can explain the experimental strength data much better than previous models. Basic material parameters such as the Poisson's ratio, short-fibre strength and critical load-transfer length were included in the equation for calculating the composite strength. Various strengthening mechanisms considered in the analysis include the metal matrix dispersion hardening by randomly oriented short fibres, the dislocation strengthening and the residual thermal stress in the fibres. This analysis may provide composite engineers with some very useful information for the design of composite materials.

Appendix: the effective fibre number N_e

The average projective fibre length on the loading direction, $\bar{l}^e_\parallel$, can be calculated in a similar way to $f(\theta)$

$$\bar{l}^e_\parallel = \frac{1}{N}\int_0^{\pi/2} l_e\cos(\theta)\, N\sin(\theta)\, d\theta = \frac{l_e}{2} \quad (A1)$$

where l_e is the effective length of the short fibre and θ is as defined before. l_e can be expressed as

$$l_e = l - l_c/2 \quad (A2)$$

where l and l_c are short-fibre length and critical load-transfer length, respectively,

Assuming that there are in total N randomly oriented fibres and the composite sample length is L, the effective number of fibres cut by a sample cross-section perpendicular to the loading direction can be calculated as

$$N_e = \frac{N\bar{l}^e_\parallel}{L} \quad (A3)$$

N can be calculated from the following equation

$$N = \frac{LAV_f}{la_f} \quad (A4)$$

where A is the sample cross-sectional area. Substituting Equations A1, A2 and A3 into Equation A4 yields

$$N_e = \frac{AV_f}{2a_f}\left(1 - \frac{l_c}{2l}\right) \quad (A5)$$

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